SUBORDINATION OF HILBERT SPACE VALUED LÉVY PROCESSES

FRED ESPEN BENTH AND PAUL KRÜHNER

ABSTRACT. We generalise multivariate subordination of Lévy processes as introduced by Barndorff-Nielsen, Pedersen, and Sato [2] to Hilbert space valued Lévy processes. The processes are explicitly characterised and conditions for integrability and martingale properties are derived under various assumptions of the Lévy process and subordinator. As an application of our theory we construct explicitly some Hilbert space valued versions of Lévy processes which are popular in the univariate and multivariate case. In particular, we define a normal inverse Gaussian Lévy process in Hilbert space as a subordination of a Hilbert space valued Wiener process by an inverse Gaussian Lévy process. The resulting process has the property that at each time all its finite dimensional projections are multivariate normal inverse Gaussian distributed as introduced in Rydberg [16].

1. INTRODUCTION

Subordination, which was first introduced by Bochner [5], has become a widely used tool to construct new Markov processes or C_0-semigroups. Barndorff-Nielsen, Pedersen and Sato [2] extended this approach to multivariate subordination of Lévy processes, i.e. subordination of d independent Lévy processes $L_1, \ldots, L_d$ with d possibly dependent subordinators $\Theta_1, \ldots, \Theta_d$. They proved that the resulting process $X(t) := (L_1(\Theta_1(t)), \ldots, L_d(\Theta_d(t)))$ is again a Lévy process and its characteristics as well as its Lévy exponent can be expressed easily in terms of properties of L and Θ. In the recent paper of Mendoza-Arriaga and Linetsky [12] multivariate subordination has been generalised to Markov processes with locally compact state spaces. Baeumer, Kovács and Meerschaert [1] treated multivariate subordination from an analytical point of view.

Peszat and Zabzcyk [14, page 62] indicate that the usual subordination procedure can be used to generate new Hilbert space valued Lévy processes. We follow their suggestion, and introduce multivariate subordination of Hilbert space valued Lévy processes. In particular, we subordinate a cylindrical Brownian motion with an inverse Gaussian process which generalises subordination of real valued Brownian motions with the same subordinator. The latter subordinated process is a so-called normal inverse Gaussian Lévy process, while the first an infinite dimensional generalization of it. As it turns out, projections of this Hilbert space valued normal inverse Gaussian Lévy process to finite dimensional subspaces become multivariate normal inverse Gaussian distributed Lévy processes (cf.

Øigard and Hanssen [13] for definition of the multivariate normal inverse Gaussian distribution). We also introduce α-stable and variance Gamma processes in infinite dimensions. Hilbert space valued Lévy processes can be applied to modeling of the spatio-temporal dynamics of weather variables like wind and temperature, and the evolution of futures prices in energy markets or forward rates in fixed-income markets. Other areas of application includes quantum physics and turbulence.

The subordinated Lévy processes can be completely characterised by the characteristics of the Lévy process and subordinator. Moreover, we analyse in detail the integrability properties of the Hilbert space valued subordinated Lévy process. Finite first and second moments of these infinite dimensional processes can be shown to exists under various mild conditions on the Lévy process and/or the subordinator process. We derive several different conditions under which (square-)integrability holds.

This paper is arranged as follows. In the second section we introduce multivariate subordination of Hilbert space valued Lévy processes and give formulas for the characteristic function and the characteristics of the subordinated process. In the third section we characterise the second order moment structure and characterise martingale property of the subordinated process. In the fourth section Hilbert space valued normal inverse Gaussian processes (and other Hilbert space valued Lévy processes) are introduced and we apply the results of the previous sections to them.

1.1. **Mathematical preliminaries.** $\mathbb{R}$, resp. $\mathbb{C}$, denotes the real, resp. the complex number, and $\mathbb{R}_+ := [0, \infty)$ (resp. $\mathbb{R}_- := (-\infty, 0]$) the non-negative (resp. non-positive) real numbers. $(\Omega, \mathfrak{A}, P)$ will always denote a probability space. If not otherwise stated, we will allways assume that our stochastic processes have càdlàg paths and work with the truncation function $\chi(x) := x 1_{\{|x| \leq 1\}}$.

Throughout this article let $d \in \mathbb{N}$, $(H_j, \langle \cdot | \cdot \rangle_j)$ be separable Hilbert space and L_j be an H_j-valued Lévy process for $j = 1, \ldots, d$ such that $L_1, \ldots, L_d$ are independent, cf. Peszat and Zabczyk [14, Section 4]. Let (b_j, Q_j, ν_j) be the characteristics of L_j (we provide a proof of the uniqueness of the characteristics in Lemma A.1) and denote the Lévy exponent of L_j by φ_j for all $j = 1, \ldots, d$, i.e. $\varphi_j : H_j \to \mathbb{C}$ such that

$$\mathrm{E} e^{i\langle L_j(t) | u \rangle} = \exp(t \varphi_j(u))$$

for any $t \in \mathbb{R}_+$, $u \in H_j$ (cf. Peszat and Zabcyzk [14, Section 4.6]). Define $L := (L_1, \ldots, L_d)$, $H := H_1 \otimes \cdots \otimes H_d$ and $\langle u | v \rangle := \sum_{j=1}^d \langle u_j | v_j \rangle_j$ for $u, v \in H$. Let (b, Q, ν) be the characteristics of L. For $\theta \in \mathbb{R}_+^d$ we define $L(\theta) := (L_1(\theta_1), \ldots, L_d(\theta_d))$. For $a \in \mathbb{R}^d$ and $u \in H$ we define $au := (a_1 u_1, \ldots, a_d u_d) \in H$. For bounded linear operators $T_1, \ldots, T_d$ on $H_1, \ldots, H_d$ we define $T_1 \times \cdots \times T_d : H \to H, u \mapsto (T_1 u_1, \ldots, T_d u_d)$ and for $a \in \mathbb{R}_+^d$ and $T := T_1 \times \cdots \times T_d$ we also define $aT := a_1 T_1 \times \cdots \times a_d T_d$.

Let Θ be a Lévy process with values in $\mathbb{R}^d$ such that Θ_j is a subordinator for all $j = 1, \ldots, d$, cf. Sato [17, Definition 21.4] or Skorokhod [18]. Let (a, c, F) be the Lévy-Khintchine triplet of Θ. Then $c = 0$ and $\int_{\{|\theta| \leq 1\}} \theta F(d\theta) < \infty$ since the paths of Θ are of bounded variation, cf. [17, Theorem 21.9]. Define $a_0 := a - \int_{\mathbb{R}_+^d} \chi(\theta) F(d\theta)$ and

$$\psi : (\mathbb{R}_- + i\mathbb{R})^d \to \mathbb{C}, s \mapsto a_0 s + \int_{\mathbb{R}_+^d} (e^{s\theta} - 1) F(d\theta).$$

From [17, Theorem 8.1] it can be seen that $Ee^{s\Theta(1)} = \exp(\psi(s))$ for any $s \in (\mathbb{R}_- + i\mathbb{R})^d$ and [17, Theorem 21.5] yields $a_0 \in (\mathbb{R}_+)^d$ and F is concentrated on $(\mathbb{R}_+)^d$.

Further unexplained notation is used as in the books of Jacod and Shiryaev [7] and Peszat and Zabczyk [14].

Remark 1.1. Like in the finite dimensional case there is a connection between the characteristics of a Lévy process and its Lévy exponent. Indeed, [14, Theorem 4.27] yields

$$\varphi_j(u) = i\langle u|b_j\rangle_j - \frac{1}{2}\langle Q_j u|u\rangle_j + \int_{H_j} \left(e^{i\langle u|x\rangle_j} - 1 - i\langle u|\chi(x)\rangle_j\right) \nu_j(dx)$$

for any $u \in H_j$ and any $j = 1, \ldots, d$. Moreover, the triplet of $L = (L_1, \ldots, L_d)$ can of course be expressed in the triplets of $L_1, \ldots, L_d$. Namely we have

$$\begin{aligned} b &= (b_1, \ldots, b_d) \\ Q &= Q_1 \times \cdots \times Q_d \\ \nu(A) &= \sum_{j=1}^{d} \nu_j^{\eta_j}(A) \end{aligned}$$

for any $A \subseteq \mathcal{B}(H)$ where η_j is the natural embedding from H_j into H, e.g. $\eta_1 : H_1 \to H, u \mapsto (u, 0, \ldots, 0)$.

As a sideremark we want to note that H is a modul over the ring $(\mathbb{R}^d, +, \cdot)$ with respect to the multiplication $(a, u) \mapsto au$ as defined above where $\cdot$ is the componentwise multiplication on $\mathbb{R}^d$. The mapping Q is an $\mathbb{R}^d$-linear mapping.

The case $d = 1$ will be of special interest in this article and after Section 3 the results for this particular case will be used only.

2. SUBORDINATED HILBERT SPACE VALUED LÉVY PROCESSES

Multivariate subordination of $\mathbb{R}^d$-valued Lévy processes has been treated in Barndorff-Nielsen, Pedersen and Sato [2]. We extend their results to Hilbert space valued processes, and define the *multivariate subordinated Lévy process*

$$X(t) := (L_1(\Theta_1(t)), \ldots, L_d(\Theta_d(t))) = L(\Theta(t)) \tag{1}$$

for any $t \geq 0$.

As we shall see in this Section, the Lévy exponent of the subordinated Lévy process can be easily expressed in the Lévy exponent of the original Lévy processes and the Laplace exponent of the subordinator, see Theorem 2.3 below. Moreover, the characteristics of the subordinated Lévy process can be expressed in terms of the characteristics of the original Lévy processes, the characteristics of the subordinators and the distribution of the original Lévy processes, see Theorem 2.4 below.

Remark 2.1. The process X has càdlàg paths because $\Theta_1, \ldots, \Theta_d$ have càdlàg paths and $L_1, \ldots, L_d$ have càdlàg paths.

Remark 2.2. Observe that the set of functions

$$\{f_u : H \to \mathbb{C}, x \mapsto e^{i\langle u|x\rangle} : u \in H\}$$

is a monotone class. Hence [6, Corollary A.4.4] yields that the law of H-valued random variables Y, Z coincide if and only if

$$\mathrm{E}(e^{i\langle u|Y\rangle}) = \mathrm{E}(e^{i\langle u|Z\rangle})$$

for any $u \in H$.

Theorem 2.3. *The process X is a Lévy process and its Lévy exponent is given by*

$$\rho : H \to \mathbb{C}, u \mapsto \psi\left((\varphi_1(u_1), \ldots, \varphi_d(u_d))\right).$$

Proof. This proof is along the lines of the proof of [2, Theorem 3.3]. Let $n \in \mathbb{N}$, $u \in H^n$ and define

$$T_n := \{\theta \in \mathbb{R}_+^{(n+1)\times d} : \theta_{k,j} < \theta_{k+1,j} \text{ for any } k = 1, \ldots, n, j = 1, \ldots, d\}.$$

Let

$$f : T_n \to \mathbb{C}, \theta \mapsto \mathrm{E}\exp\left(i\sum_{k=1}^{n}\langle u_k|(L(\theta_{k+1}) - L(\theta_k))\rangle\right).$$

Independence of the coordinates of L, independence of the increments of L and [14, Theorem 4.27] yield

$$
\begin{aligned}
f(\theta) &= \prod_{k=1}^{n}\prod_{j=1}^{d}\exp((\theta_{k+1,j} - \theta_{k,j})\varphi_j(u_{k,j})) \\
&= \exp\left(\sum_{k=1}^{n}\sum_{j=1}^{d}(\theta_{k+1,j} - \theta_{k,j})\varphi_j(u_{k,j})\right)
\end{aligned}
$$

for any $\theta \in T_n$. Since L and Θ are independent we get

$$
\begin{aligned}
&\mathrm{E}\exp\left(i\sum_{k=1}^{n}\langle u_k|L(\Theta(t_{k+1})) - L(\Theta(t_k))\rangle\right) \\
&= \mathrm{E}f((\Theta_j(t_k))_{k\in\{1,\ldots,n+1\},j\in\{1,\ldots,d\}}) \\
&= \mathrm{E}\left(\exp\left(\sum_{k=1}^{n}\sum_{j=1}^{d}(\Theta_j(t_{k+1}) - \Theta_j(t_k))\varphi_j(u_{k,j})\right)\right) \\
&= \prod_{k=1}^{n}\mathrm{E}\left(\exp\left(\sum_{j=1}^{d}(\Theta_j(t_{k+1}) - \Theta_j(t_k))\varphi_j(u_{k,j})\right)\right) \\
&= \prod_{k=1}^{n}\exp\left((t_{k+1} - t_k)\psi((\varphi_j(u_{k,j}))_{j=1,\ldots,d})\right) \\
&= \prod_{k=1}^{n}\exp\left((t_{k+1} - t_k)\rho(u_k)\right)
\end{aligned}
$$

for any $0 \le t_1 < \cdots < t_{n+1}$. Now it follows that X is a Lévy process.

Moreover, for $n = 1$, $t_2 = 1$, $t_1 = 0$ we have

$$\mathrm{E}\exp\left(i\langle u|L(\Theta(1))\rangle\right) = \exp\left(\rho(u)\right)$$

which is the claimed formula. □

We are now ready to compute the characteristics of the multivariate subordinated Lévy process X.

Theorem 2.4. *We have $\int_{\mathbb{R}_+^d} |\mathrm{E}(\chi(L(\theta)))|F(d\theta) < \infty$. Define*

$$\beta = a_0 b + \int_{\mathbb{R}_+^d} \mathrm{E}(\chi(L(\theta)))F(d\theta),$$

$$\Gamma = a_0 Q \quad \text{and}$$

$$\mu(A) = \sum_{j=1}^d a_{0,j} \nu_j^{\eta_j}(A) + \int_{\mathbb{R}_+^d} P^{L(t)}(A)F(d\theta)$$

for any Borel-sets $A \subseteq H$. Then (β, Γ, μ) is the characteristics of X.

Proof. Define the measure

$$\widetilde{\mu}(A) := \int_{\mathbb{R}_+^d} P^{L(\theta)}(A)F(d\theta)$$

for any Borel-sets $A \subseteq H$. Observe that for any measurable function $f : H \to \mathbb{R}$ which is positive or $\widetilde{\mu}$-integrable we have

$$\int_H f d\widetilde{\mu} = \int_{\mathbb{R}_+^d} \mathrm{E}(f(L(\theta)))F(d\theta).$$

By Lemma A.20 there is $C > 1$ such that $|\mathrm{E}\chi(L(\theta))| \leq |\theta|C$ for any $\theta \in \mathbb{R}_+^d$. Thus $\theta \mapsto |\mathrm{E}\chi(L(\theta))|$ is bounded by $(1 \wedge |\theta|)C$. Hence [17, Theorem 21.5] yields that it is F-integrable which is the first part of the claim. Theorem 2.3 yields that the Lévy exponent of X is given by

$$\rho(u) := \psi((\varphi_j(u_j))_{j=1,\dots,d})$$

for any $u \in H$. Then

$$\begin{aligned}
\rho(u) &= \psi((\varphi_j(u_j))_{j=1,\dots,d}) \\
&= \sum_{j=1}^d a_{0,j} \varphi_j(u_j) + \int_{\mathbb{R}_+^d} \left(e^{\sum_{j=1}^d \theta_j \varphi_j(u_j)} - 1 \right) F(d\theta)
\end{aligned}$$

for any $u \in H$. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}_+^d} \left(e^{\sum_{j=1}^d \theta_j \varphi_j(u_j)} - 1 \right) F(d\theta) \\
&= \int_{\mathbb{R}_+^d} \left(\mathrm{E}\left(\exp\left(i\langle L(\theta)|u\rangle \right) \right) - 1 - i\langle \mathrm{E}\chi(L(\theta))|u\rangle \right) F(d\theta) + i\langle \gamma|u\rangle \\
&= \int_{\mathbb{R}_+^d} \mathrm{E}\left(\exp\left(i\langle L(\theta)|u\rangle \right) - 1 - i\langle \chi(L(\theta))|u\rangle \right) F(d\theta) + i\langle \gamma|u\rangle
\end{aligned}$$

where $\gamma = \int_{\mathbb{R}_+^d} \mathrm{E}\chi(L(\theta))F(d\theta)$ for any $u \in H$. Let $u \in H$ and define

$$f : H \to \mathbb{R}_+, x \mapsto \exp\left(i\langle u|x\rangle \right) - 1 - i\langle \chi(x)|u\rangle.$$

Lemma A.21 yields $g(\theta) := \mathrm{E}|f(L(\theta))| \le |\theta| C_2$ for some $C_2 > 0$ and any $\theta \in \mathbb{R}_+^d$. Since g is positive and bounded by some constant C_3, we have

$$
\begin{aligned}
\int_H |f(x)| \widetilde{\mu}(dx) & = \int_{\mathbb{R}_+^d} g(\theta) F(d\theta) \\
& \le \int_{\mathbb{R}_+^d} (1 \wedge |\theta|) F(d\theta)(C_2 \vee C_3) \\
& < \infty
\end{aligned}
$$

Thus f is $\widetilde{\mu}$-integrable. Hence we have

$$
\int_{\mathbb{R}_+^d} \left(e^{\sum_{j=1}^d \theta_j \varphi_j(u_j)} - 1 \right) F(d\theta) = \int_H f(x) \widetilde{\mu}(dx) + i \langle \gamma | u \rangle.
$$

[14, Theorem 4.27] implies that the characteristics can be read from the representation

$$
\rho(u) = \sum_{j=1}^d a_{0,j} \varphi_j(u_j) + \int_H f(x) \widetilde{\mu}(dx) + i \langle \gamma | u \rangle.
$$

Hence, the proof is complete. $\qquad\square$

3. PROBABILISTIC FEATURES OF SUBORDINATED LÉVY PROCESSES

In this section we want to investigate the probabilistic features of the subordinated Lévy process $X(t) = L(\Theta(t))$, i.e. we give necessary and sufficient conditions for X to have finite first or second moment and provide formulas for those moments. Thanks to [14, Section 4.9] one can characterise finiteness of the second moment of X completely in terms of moments of L and Θ. It turns out that square integrability of L and Θ are sufficient and essentially necessary for square integrability of X if L is not a martingale. If L is a square-integrable martingale, then integrability of Θ is sufficient and essentially necessary to ensure that X is square integrable, cf. Theorem 3.7 below. We also show that X is integrable if L and Θ are integrable where we make use of several results collected in the Appendix, cf. Theorem 3.10 below. If L is a square-integrable martingale, then it is sufficient that $\sqrt{|\Theta(1)|}$ is integrable, cf. Theorem 3.9 below. However, the authors do not know if, under the assumption that L is square-integrable, the integrability of $\sqrt{|\Theta(1)|}$ is necessary for integrability of X. Corollary 3.11 shows that this is true if L is a cylindrical Brownian motion. If L is a martingale but not square-integrable, then it is possible that integrability of $\sqrt{|\Theta(1)|}$ is not sufficient to ensure integrability of X as we will show in Proposition 4.8 at the end of Section 4.2.

Remark 3.1. Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be a filtration such that $L(t) - L(s)$ is independent of $\mathcal{F}_s$. Then the following statements are equivalent.

- L is a $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-martingale.
- L is a martingale w.r.t. its own (right-continuous) filtration.
- L is mean zero, i.e. L has finite expectation and $\mathrm{E}L(t) = 0$ for any $t \in \mathbb{R}_+$.
- $\int_{\{|x|>1\}} |x| \nu(dx) < \infty$ and $0 = b + \int_{\{|x|>1\}} x \nu(dx)$.

Definition 3.2. Let Y be any H-valued random variable with finite second moment. Then the *covariance operator* $\mathrm{Cov}(Y)$ of Y is defined by the equation

$$\langle \mathrm{Cov}(Y)x|y\rangle = \mathrm{E}(\langle Y - \mathrm{E}Y|x\rangle\langle Y - \mathrm{E}Y|y\rangle)$$

for any $x, y \in H$.

We first recall some properties of square integrable Lévy processes.

Proposition 3.3. *The Lévy process L is square integrable if and only if $\int_H |x|^2\nu(dx) < \infty$. If L is square integrable, then*

- $\mathrm{E}(L(t)) = t\left(b + \int_{|x|>1} x\nu(dx)\right)$,
- $\mathrm{E}(|L(t) - \mathrm{E}L(t)|^2) = t\left(\mathrm{Tr}(Q) + \int_H |x|^2\nu(dx)\right)$,
- $M(t) := L(t) - t\mathrm{E}L(1)$ *is a mean zero and square integrable Lévy process with the characteristics $(b - \mathrm{E}L(1), Q, \nu)$ and*
- $\langle \mathrm{Cov}(L(t))x|y\rangle = \langle Qx|y\rangle + \int_H \langle x|z\rangle\langle y|z\rangle\nu(dz)$ *for any $x, y \in H$.*

Proof. See [14, Theorem 4.47 and Theorem 4.49]. □

Remark 3.4. The covariance operator of L can, of course, be expressed in the covariance operators of $L_1, \ldots, L_d$. We have

$$\mathrm{Cov}(L(1)) = \mathrm{Cov}(L_1(1)) \times \cdots \times \mathrm{Cov}(L_d(1)).$$

We first aim at characterising square integrability of the subordinated process X, see Theorem 3.7 below. The proof is devided into three parts where the next two lemmas each contain a part. It is essentially necessary that L is square integrable for X being square integrable. However, square integrability of the multivariate subordinator Θ is only needed if L is not a martingale. If L is a square integrable martingale, then integrability for Θ is sufficient to ensure that X is square integrable. This is the statement of the next Lemma.

Lemma 3.5. *Let L be mean zero and square integrable and Θ be integrable. Then X is mean zero and square integrable and*

$$\mathrm{Cov}(X(1)) = \mathrm{E}(\Theta(1))\mathrm{Cov}(L(1)).$$

Proof. Let $g(\theta) := \mathrm{E}(|L(\theta)|^2)$ for any $\theta \in \mathbb{R}_+^d$. Proposition 3.3 yields

$$g(\theta) = \sum_{j=1}^d \theta_j \mathrm{E}(|L_j(1)|^2)$$

for any $\theta \in \mathbb{R}_+^d$. By conditioning on Θ we get

$$
\begin{aligned}
\mathrm{E}(|X(t)|^2) &= \mathrm{E}g(\Theta(t)) \\
&= \sum_{j=1}^d \mathrm{E}\Theta_j(t)\mathrm{E}(|L_j(1)|^2) \\
&< \infty
\end{aligned}
$$

for any $t \geq 0$. Thus X is square integrable. Conditioning on Θ yields $\mathrm{E}X(t) = 0$ for any $t \geq 0$ and

$$
\begin{aligned}
\langle \mathrm{Cov}(X(1))a|b \rangle &= \mathrm{E}(\langle \langle X(1))|a \rangle \langle X(1)|b \rangle) \\
&= \sum_{j=1}^{d} \mathrm{E}(\Theta_j(1)) \langle \mathrm{Cov}(L_j(1))a_j|b_j \rangle \\
&= \left\langle \mathrm{E}(\Theta(1))\mathrm{Cov}(L(1))a \middle| b \right\rangle
\end{aligned}
$$

for any $a, b \in H$. $\qquad \square$

If L is square integrable but not a martingale, then Θ has to be square integrable in order to ensure that X is square integrable. Theorem 3.7 below will show that square integrability of Θ is essentially necessary to ensure square integrability of X.

Lemma 3.6. *Let L and Θ be square integrable. Then X is square integrable,*

$$
\mathrm{E}(X(1)) = \mathrm{E}\Theta(1)\mathrm{E}L(1) \quad \text{and}
$$

$$
\mathrm{Cov}(X(1)) = \mathrm{E}(\Theta(1))\mathrm{Cov}(L(1)) + \sum_{i,j=1}^{d} \mathrm{Cov}(\Theta(1))_{i,j}(\mathrm{E}L_i(1)) \otimes (\mathrm{E}L_j(1))
$$

where $x \otimes y : H \to H, z \mapsto \langle x|z \rangle y$ for any $x, y \in H$.

Proof. This follows easily by conditioning on the process Θ. $\qquad \square$

We can now state the characterisation of square integrability of X.

Theorem 3.7. *X is square integrable if and only if X_j is square integrable for all $j = 1, \ldots, d$. Let $j \in \{1, \ldots, d\}$. Then X_j is square integrable if and only if any of the following statements hold.*

(1) L_j and Θ_j are square integrable.
(2) L_j is mean zero and square integrable and Θ_j is integrable.
(3) $\Theta_j = 0$ a.s.
(4) $L_j = 0$ a.s.

Moreover, X_j is mean zero and square integrable if and only if (2), (3) or (4) holds. If (1) holds for any $j = 1, \ldots, d$, then

$$
\mathrm{E}(X(1)) = \mathrm{E}\Theta(1)\mathrm{E}L(1) \quad \text{and}
$$

$$
\mathrm{Cov}(X(1)) = \mathrm{E}(\Theta(1))\mathrm{Cov}(L(1)) + \sum_{i,j=1}^{d} \mathrm{Cov}(\Theta(1))_{i,j}(\mathrm{E}L_i(1)) \otimes (\mathrm{E}L_j(1))
$$

where $x \otimes y : H \to H, z \mapsto \langle x|z \rangle y$ for any $x, y \in H$. If (2) holds for any $j = 1, \ldots, d$, then

$$
\mathrm{Cov}(X(1)) = \mathrm{Cov}(L(1))\mathrm{E}(\Theta(1)).
$$

Proof. The first statement follows directly from the equation $|X(1)|^2 = \sum_{j=1}^{d} |X_j(1)|^2$. The formulas at the end of the Theorem follow from the two previous Lemmas. For the characterisation of square integrability of X_j we can assume w.l.o.g. that $d = 1$.

The if part follows from the two previous Lemmas. Assume that X is square integrable. Let (β, Γ, μ) be the characteristics of X as given in Theorem 2.4. Proposition 3.3 yields

$$
\begin{aligned}
\infty \; > \; & \int_H |x|^2 \mu(dx) \\
= \; & \int_H |x|^2 (a_0 \nu)(dx) + \int_0^\infty \mathrm{E}(|L(\theta)|^2) F(d\theta).
\end{aligned}
$$

Thus $\int_0^\infty \mathrm{E}(|L(\theta)|^2) F(d\theta) < \infty$ and $\int_H |x|^2 (a_0 \nu)(dx) < \infty$.

Case 1: L is not square integrable. Then Proposition 3.3 implies that $\int_H |x|^2 \nu(dx) = \infty$. Thus $F = 0$ and $a_0 = 0$. Hence $\Theta = 0$ a.s. which is statement (3).

Case 2: L is square integrable. Let $v := \mathrm{E}(|L(1) - \mathrm{E}L(1)|^2)$ and $m := \mathrm{E}L(1)$. Then $\mathrm{E}(|L(\theta)|^2) = \theta v + \theta^2 |m|^2$ for any $\theta \geq 0$. Hence we have

$$
\int_0^\infty \theta v F(d\theta) \; < \; \infty \quad \text{and}
$$
$$
\int_0^\infty \theta^2 |m|^2 F(d\theta) \; < \; \infty,
$$

Case 2.1: $m \neq 0$. Then $\int_0^\infty \theta^2 F(d\theta) < \infty$. Hence [17, Corollary 25.8] yields that Θ is square integrable.

Case 2.2: $m = 0, v \neq 0$. Then L is mean zero and $\int_0^\infty \theta F(d\theta) < \infty$. Hence [17, Corollary 25.8] yields that Θ is integrable. Thus we have statement (2).

Case 2.3: $m = 0, v = 0$. Since $0 = v = \mathrm{E}(|L(1)|^2)$ we have $L = 0$ a.s.

Lemma 3.5 yields that if (2), (3) or (4) holds, then X is mean zero and square integrable. If X is mean zero and square integrable and (1) holds, then we have

$$
0 = \mathrm{E}(X(1)) = \mathrm{E}\Theta(1)\mathrm{E}L(1).
$$

Thus $\mathrm{E}\Theta(1) = 0$ which yields (3) or $\mathrm{E}L(1) = 0$ which implies (2). $\qquad\square$

Theorem 3.7 above is a complete characterisation of the second order structure of the Lévy process X. However, there are Lévy processes without finite second moment (e.g. see Theorem 4.7 below). In that case the first order structure and the martingale property are still interesting. We now develop necessary and sufficient conditions for the existence of a first moment (cf. Theorem 3.10) and we give a condition that suffices to show that X is a martingale. Corollary 3.11 is a restatement of Theorem 3.10 for the special case that L is a Brownian motion without drift.

Lemma 3.8. *Let L and Θ be integrable. Then X is integrable and*

$$
\mathrm{E}X(1) = \mathrm{E}\Theta(1)\mathrm{E}L(1).
$$

Proof. If X is integrable, then conditioning on Θ yields the formula.

Let f be the growth function of L, cf. Definition A.9. Then Lemma A.18 yields that there is $C > 0$ and $f(\theta) \leq 1 + C|\theta|$ for any $\theta \in \mathbb{R}_+^d$. Lemma A.11 yields that X is integrable if $f(\Theta(1))$ is integrable. However,

$$
\mathrm{E}f(\Theta(1)) \leq 1 + C\mathrm{E}|\Theta(1)| < \infty.
$$

$\qquad\square$

We have seen that the martingale property of L allows to put weaker assumptions on Θ to ensure that X is square integrable. If L is a square integrable martingale, then similar as before a weaker assumption than in Lemma 3.8 on Θ is sufficient to ensure integrability of X.

Theorem 3.9. *Let L be a square integrable martingale and assume that $\sqrt{|\Theta(1)|}$ is integrable (or equivalently $\int_{|\theta|>1} \sqrt{|\theta|} F(d\theta) < \infty$). Then X is integrable and mean zero.*

Proof. Proposition A.12 yields $E(\sqrt{|\Theta(1)|}) < \infty$ if and only if $\int_{|\theta|>1} \sqrt{|\theta|} F(d\theta) < \infty$.

Let f be the growth function of L in the sense of Defintion A.9. Then Lemma A.17 yields that there is $C > 0$ such that $f(\theta) \leq C\sqrt{|\theta|}$ for any $\theta \in \mathbb{R}_+^d$. Thus

$$E f(\Theta(1)) \leq C E \sqrt{|\Theta(1)|} < \infty.$$

Lemma A.11 yields that X is integrable. Moreover, $E(X(1)|\Theta) = \Theta(1)E(L(1)) = 0$. Thus X is mean zero. $\qquad\square$

Theorem 3.10. *Let Θ be non trivial, i.e. $P(\Theta \neq 0) > 0$. Then X is integrable if and only if L is integrable and $\int_{|\theta|>1} E(|L(\theta)|)F(d\theta) < \infty$. If L and Θ (and hence X) are integrable, then*

$$EX(1) = EL(1)E\Theta(1).$$

If X_j is integrable but Θ_j is not integrable, then X_j is mean zero where $j \in \{1, \ldots, d\}$.

Proof. Let (β, Γ, μ) be the characteristics of X as given in Theorem 2.4 and let f be the growth function of L, cf. Definition A.9. We have

$$\int_{\{|x|>1\}} |x|\mu(dx) = \int_{\{|x|>1\}} |a_0 x|\nu(dx) + \int_{\mathbb{R}_+^d} E(|L(\theta)|1_{|L(\theta)|>1})F(d\theta).$$

$\Rightarrow$: Let X be integrable. Then Proposition A.12 yields $\int_{\{|x|>1\}} |x|\mu(dx) < \infty$. Thus Proposition A.12 implies that L is integrable.

$\Leftarrow$: Let L be integrable and $\int_{|\theta|>1} E(|L(\theta)|)F(d\theta) < \infty$. Then

$$\int_{|\theta|>1} f(\theta)F(d\theta) = \int_{|\theta|>1} E|L(\theta)|F(d\theta) < \infty$$

where f denotes the growth function of L. Proposition A.12 yields $E f(\Theta(1)) < \infty$. Lemma A.11 yields the first claim.

Lemma 3.8 yields the formula for the moment above.

Now let $j \in \{1, \ldots, d\}$ and assume that X_j is integrable but Θ_j is not. We have already shown that this implies that L_j is integrable. Let $g(\theta) := EL_j(\theta) = \theta EL_j(1)$ for any $\theta \in \mathbb{R}_+$. Thus

$$EX_j(1) = E(E(L_j(\Theta_j(1))|\Theta)) = Eg(\Theta(1)) = E(\Theta_j(1)EL_j(1)).$$

We see from that equation that $\Theta_j(1)EL_j(1)$ is integrable. Since $\Theta_j(1)$ is not integrable we conclude that $EL_j(1) = 0$. Hence $EX_j(1) = 0$. $\qquad\square$

We are especially interested in the case that L is a Gaussian Lévy process (i.e. a cylindrical Brownian motion). Then the martingale property of X can be characterised easily in terms of a moment condition of Θ.

Corollary 3.11. *Assume that L is Gaussian and mean zero and assume that $\mathrm{Tr}(Q_j) \neq 0$ for all $j \in \{1, \dots, d\}$. Then X is integrable if and only if $\sqrt{|\Theta(1)|}$ is integrable. In that case X is mean zero.*

Proof. Theorem 3.9 implies the if part and the last statement. Let X be integrable. Theorem 3.10 yields $\int_{|\theta|>1} \mathrm{E}|L(\theta)| F(d\theta) < \infty$. We also have

$$
\begin{aligned}
\mathrm{E}|L(\theta)| &\geq \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \mathrm{E}|L_j(\theta_j)| \\
&= \frac{1}{\sqrt{d}} \sum_{j=1}^{d} \theta_j^{1/2} \mathrm{E}|L_j(1)| \\
&\geq \frac{1}{\sqrt{d}} \sqrt{|\theta|} \min\{\mathrm{E}|L_j(1)| : j = 1, \dots, d\} \\
&= \sqrt{|\theta|} C
\end{aligned}
$$

for any $\theta \in \mathbb{R}_+^d$ where $C := \frac{\min\{\mathrm{E}|L_j(1)| : j=1,\dots,d\}}{\sqrt{d}} > 0$. Hence

$$
\int_{|\theta|>1} \sqrt{|\theta|} F(d\theta) < \infty.
$$

Proposition A.12 yields $\sqrt{|\Theta(1)|}$ is integrable. $\qquad\square$

Gaussian Lévy processes L will play a main role when defining some explicit classes of subordinated Lévy processes, which is the topic of the next Section.

4. EXAMPLES AND APPLICATION

In this Section we construct three classes of subordinated Lévy processes, extending the popular uni/multi-variate normal inverse Gaussian, α-stable and variance Gamma Lévy processes.

4.1. **Hilbert space valued normal inverse Gaussian process.** Multivariate normal inverse Gaussian distributions (MNIG-distributions) have been first introduced in [16]. These distributions can, of course, also be generated from a multivariate Brownian motion and an inverse Gaussian process by subordination, i.e. the subordinated Brownian motion is a process where its marginal distributions are MNIG. We generalise this approach to construct Hilbert space-valued normal inverse Gaussian (HNIG) processes.

Definition 4.1. A Lévy process Y is an *HNIG-process* if there are $s, c \in \mathbb{R}_+$, $b \in H$ and a positive semi-definite trace class operator Q on H such that its Lévy exponent is given by

$$
\rho : H \to \mathbb{C}, u \mapsto s\left(c - \sqrt{c^2 + \langle Qu|u\rangle - i2\langle u|b\rangle}\right)
$$

where $\sqrt{\cdot}$ denotes the main branch of the root function. Here, (s, c, b, Q) are the *parameters of the HNIG-process Y*. A *degenerate HNIG-process* is an HNIG-process where its second parameter is 0, i.e. there are $s \in \mathbb{R}_+$, $b \in H$ and a positive semi-definite trace class operator Q on H such that $(s, 0, b, Q)$ are the parameters of Y.

Let us start with the construction of non-degenerate HNIG processes and discuss some of their properties.

Theorem 4.2. *Let $s, c \in \mathbb{R}_+$, $c \neq 0$, $b \in H$ and Q a positive semi-definite trace class operator on H. Then there is an HNIG-process Y with parameters (s, c, b, Q). The characteristics (β, Γ, μ) of Y are given by*

$$\beta = \frac{sb}{c} - \int_{|x|>1} x\mu(dx),$$

$$\Gamma = 0 \quad and$$

$$\mu(A) = \int_0^\infty \Phi_\theta(A) \frac{s}{\sqrt{2\pi\theta^3}} e^{-c^2\theta/2} d\theta$$

for any Borel set $A \subseteq H$ where Φ_θ denotes the Gaussian measure on H with mean θb and covariance operator θQ. Moreover, $\mathrm{E}Y(1) = \frac{s}{c}b$ and $\mathrm{Cov}Y(1) = \frac{sb\otimes b}{c^3} + \frac{s}{c}Q$. If $b = 0$, then $\beta = 0$ and Y is symmetric. The distribution of $TY(t)$ is MNIG in the sense of [3, Section 10.5] for any bounded linear operator T from H to $\mathbb{R}^n$ and any $n \in \mathbb{N}$.

Remark 4.3. In the theorem above the requirement $c \neq 0$ is not needed to ensure existence. However, the resulting degenerate HNIG-process will behave differently, cf. Proposition 4.4 below.

Proof of Theorem 4.2. Let L be a Brownian motion with drift b and covariance operator Q. Let Θ be an inverse Gaussian process with parameters s, d, i.e. it is a pure-jump subordinator and its Lévy measure is given by

$$F(d\theta) = \frac{s}{\sqrt{2\pi\theta^3}} e^{-c^2\theta/2} 1_{\{\theta>0\}} d\theta,$$

cf. [3, Example 7.25]. Then its Laplace exponent is given by

$$\psi : \mathbb{R}_- + i\mathbb{R} \to \mathbb{C}, v \mapsto s\left(c - \sqrt{c^2 - 2v}\right)$$

where $\sqrt{\cdot}$ denotes the main branch of the root function. Theorem 2.3 yields that the Lévy exponent of the Lévy process $X(t) := L(\Theta(t))$ is

$$\rho : H \to \mathbb{C}, u \mapsto s\left(c - \sqrt{c^2 + \langle Qu|u\rangle - i2\langle u|b\rangle}\right).$$

Theorem 2.4 yields that (β, Γ, μ) as defined as above is the characteristics of X. Theorem 3.7 implies that X is square integrable and that its expectation and its covariance operator are given as above. Let $n \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $T : H \to \mathbb{R}^n$ be bounded and linear. Then $T(X(t)) = (T \circ L)(\Theta(t))$. $W := (T \circ L)$ is a Gaussian process on $\mathbb{R}^n$ with drift Tb and covariance operator TQT^* where T^* denotes the dual operator of T. Hence $TX(t) = W(\Theta(t))$ and consequently its distribution is MNIG, cf. [3, Section 10.5].

Let Y be any HNIG-process with parameters (s, c, b, Q). Then Remark 2.2 yields that X and Y have the same distribution and hence they have the same moments. Since X and Y have the same characteristic function they have the same characteristics. $\quad\square$

In order to construct degenerate HNIG-processes we use a different subordinator, namely 0.5-stable subordinator. Subordination of Brownian motion with an α-stable subordinator will be investigated in section 4.2 in more detail.

Proposition 4.4. *Let $s \in \mathbb{R}_+$, $b \in H$ and Q a positive semi-definite trace class operator on H. Then there is an HNIG-process C with parameters $(s, 0, b, Q)$. C is not integrable.*

Proof. Let L be a Brownian motion with drift b and covariance operator Q. Let $s \geq 0$ and Θ be the 0.5-stable subordinator with Lévy measure

$$F(d\theta) = s \frac{\theta^{-1.5}}{\Gamma(-0.5)} d\theta.$$

Then its Laplace exponent is given by

$$\psi : \mathbb{R}_- + i\mathbb{R} \to \mathbb{C}, v \mapsto \begin{cases} s \exp(-0.5 \mathrm{Log}(-v)) & v \neq 0, \\ 0 & v = 0 \end{cases}$$

where Log denotes the main branch of the logarithm. Theorem 2.3 yields that the Lévy exponent of the Lévy process $X(t) := L(\Theta(t))$ is given by

$$\rho(u) = \psi(\varphi(u)) = s \sqrt{-\varphi(u)}$$

as desired. Observe that $\sqrt{\Theta(1)}$ is not integrable. Hence Corollary 3.11 yields the claim if $b = 0$ and Theorem 3.10 yields the claim if $b \neq 0$. $\qquad\square$

Proposition 4.5. *Let Y be a process on H. Then Y is a HNIG-process if and only if TY is an MNIG-process for every finite dimensional operator T on H.*

Proof. This can be simply read from the characteristic function. $\qquad\square$

4.2. α-stable Hilbert space valued Lévy processes.

Stable Lévy processes have been studied extensively and used in mathematical finance. We refer the reader to the book of Sato [17, Chapter 3] for reference. In this section we will investigate some properties of symmetric stable Lévy processes and construct some of them, see Theorem 4.7 below. Here again we make use of subordination and generate them from Brownian motion. Like the finite dimensional case integrability properties of symmetric stable Lévy processes are related to the index of the process. Many other properties can be derived as in the finite dimensional case, cf. Sato [17, Chapter 3].

We also want to point out that CGMY processes (cf. [11]) can be constructed by subordinating a Brownian motion with drift with an α-stable subordinator. This can be easily generalised to subordination of Hilbert space valued Brownian motions.

Let us first recall the definition of strictly α-stable processes.

Definition 4.6. *Let $\alpha \in \mathbb{R}_+$. A stochastic process Y is a strictly α-stable process if $Y(t^\alpha)$ and $tY(1)$ have the same distribution for any $t \in \mathbb{R}_+$.*

An explicit construction of stable Hilbert space valued Lévy processes has been taken out in [14, Example 4.38]. We make use of this construction and discuss some properties of them.

Theorem 4.7. *For each $\alpha \in (0, 2]$ and each positive semi-definite trace class operator $Q \neq 0$ on H there is a symmetric H-valued strictly α-stable Lévy process Y with Lévy exponent*

$$\rho : H \to \mathbb{C}, u \mapsto -\langle Qu | u \rangle^{\alpha/2}.$$

Such an strictly α-stable process is square integrable if and only if $\alpha = 2$ and it is integrable and mean zero if and only if $\alpha > 1$.

Let Y be a symmetric strictly α-stable Lévy process which is non-trivial, i.e. $P(Y \neq 0) > 0$. Then $\alpha \in (0, 2]$. Moreover, there is a symmetric continuous function $f : S_H \to \mathbb{R}_+$ such that the characteristic exponent of Y is given by

$$\rho : H \to \mathbb{C}, u \mapsto -|u|^\alpha f\left(\frac{u}{|u|}\right)$$

where $S_H := \{x \in H : |x| = 1\}$ denotes the sphere in H.

Proof. Let $\alpha \in (0, 2)$ and Q be a positive definite trace class operator on H. Let L be a mean zero Gaussian Lévy process with covariance operator $2Q$. Let Θ be an $\alpha/2$-stable subordinator. Then its Laplace exponent is given by

$$\psi : \mathbb{R}_- + i\mathbb{R} \to \mathbb{C}, s \mapsto \begin{cases} -\exp(\alpha/2\mathrm{Log}(-s)) & \text{if } s \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

where Log denotes the main branch of the logarithm (cf. [4, page 73]). Hence Theorem 2.3 yields that the characteristic function of $X(t) := L(\Theta(t))$ is given by

$$\rho : H \to \mathbb{C}, u \mapsto -\langle Qu|u\rangle^{\alpha/2}.$$

We have $X(t^\alpha) = L(t^{\alpha/2}\Theta(1)) = tX(1)$ for any $t \in \mathbb{R}_+$. Hence X is a symmetric H-valued strictly α-stable Lévy process. If $\alpha = 2$, then $X = L$ and hence it is square integrable. Theorem 3.7 implies that X is not square-integrable if $\alpha \neq 2$. Corollary 3.11 yields that X is integrable if and only if $\alpha > 1$.

Now let $\alpha \in \mathbb{R}_+$ be arbitrary and Y be a strictly α-stable non-trivial Lévy process. Then there is $u \in H$ such that $P(\langle u|Y\rangle = 0) \neq 0$. [17, Theorem 13.15] applied to the strictly α-stable process $\langle u|Y\rangle$ yields that $\alpha \in (0, 2]$. Let ρ be the Lévy exponent of Y and define $f := -\rho|_{S_H}$. Let $u \in H \backslash \{0\}$ and define $t := |u|$ and $v := u/t \in S_H$. Then

$$\exp(\rho(u)) = Ee^{i\langle u|Y(1)\rangle} = Ee^{i\langle tv|Y(1)\rangle} = E^{i\langle v|Y(t^\alpha)\rangle} = \exp(t^\alpha f(v)).$$

Thus $\rho(u) = t^\alpha f(v)$. Since Y is symmetric ρ is real valued and so is f. $\mathrm{Re}(\rho)$ is bounded by 0 because the characteristic function of Y is bounded by 1. Hence $f(v) \in \mathbb{R}_+$ for any $v \in S_H$. Symmetry of f follows from symmetry of ρ which follows from symmetry of Y. $\qquad\square$

Proposition 4.8. *Let $\alpha \in (1, 2)$ and L be an integrable strictly α-stable Lévy process such that L is non-trivial, i.e. $P(L \neq 0) > 0$. Then X is integrable if and only if $|\Theta(1)|^{1/\alpha}$ is integrable.*

Proof. Let f be the growth function of L. Then $f(\theta) = \mathrm{E}|L(\theta)| = \sum_{j=1}^d \theta_j^{1/\alpha}\mathrm{E}|L_j(1)|$. Thus theorem 3.10 yields that X is integrable if and only if $\int_{\mathbb{R}_+^d} |\theta|F(d\theta) < \infty$. Proposition A.12 implies the claim. $\qquad\square$

4.3. **Hilbert space valued variance Gamma process.** Variance Gamma processes have been introduced by Madan and Seneta [10] and a multivariate version have been introduced by the same authors. Since their introduction, they have been used extensively in financial modelling (see e.g. [9]). Univariate Variance Gamma processes can be constructed as a difference of two independent Gamma processes or by subordinating a Brownian motion with a Gamma process. The latter approach can be easily generalised to Hilbert space valued Lévy processes which we do in this section. Theorem 4.10 below contains an analysis of Hilbert space valued Variance Gamma processes (HVG) and a construction of those processes is taken out in the proof of this theorem.

Definition 4.9. A Lévy process Y is a *Hilbert space valued variance gamma process* or *HVG-process* if there are $a \in \mathbb{R}_+$, $b \in H$ and a positive semi-definite trace class operator Q on H such that its Lévy exponent is given by

$$\rho : H \to \mathbb{C}, u \mapsto, a\mathrm{Log}(1 + 1/2\langle Qu|u\rangle - i\langle b|u\rangle)$$

where Log denotes the main branch of the logarithm. (a, b, Q) are the *parameters of the HVG-process Y*

Theorem 4.10. *Let $a \in \mathbb{R}_+$, $b \in H$ and Q a positive semi-definite trace class operator on H. Then there is an HVG-process Y with parameters (a, b, Q). The characteristics (β, Γ, μ) of Y are given by*

$$\begin{aligned}
\beta &= ab - \int_{|x|>1} x\mu(dx), \\
\Gamma &= 0 \ \text{and} \\
\mu(A) &= \int_0^\infty \Phi_t(A)at^{-1}e^{-t}dt
\end{aligned}$$

for any Borel set $A \subseteq H$ where Φ_t denotes the Gaussian measure on H with mean tb and covariance operator tQ. Moreover, $\mathrm{E}Y(1) = ab$ and $\mathrm{Cov}Y(1) = ab \otimes b + aQ$. $\langle u|Y\rangle$ is a variance gamma process for any $u \in H$. If $b = 0$, then $\beta = 0$ and Y is symmetric.

Proof. Let L be a Brownian motion with drift b and covariance operator Q. Let Θ be a gamma process with parameters $(a, 1)$, i.e. it is a pure-jump subordinator and its Lévy measure is given by

$$F(d\theta) = a\theta^{-1}e^{-\theta}1_{\{\theta>0\}}d\theta,$$

cf. [4, page 73]. Then its Laplace exponent is given by

$$\psi : \mathbb{R}_- + i\mathbb{R} \to \mathbb{C}, v \mapsto a\mathrm{Log}(1 - s)$$

where Log denotes the main branch of the logarithm. Theorem 2.3 yields that the Lévy exponent of the Lévy process $X(t) := L(\Theta(t))$ is

$$\rho : H \to \mathbb{C}, u \mapsto, a\mathrm{Log}(1 + 1/2\langle Qu|u\rangle - i\langle b|u\rangle).$$

Theorem 2.4 yields the specific form of the characteristics of X. Theorem 3.7 yields that X is square integrable and that its expectation and its covariance operator are given as above. Let $u \in H$. Then $\langle u|X(t)\rangle = \langle u|L\rangle(\Theta(t))$. $W := \langle u|L\rangle$ is a Gaussian Lévy process on $\mathbb{R}$ with drift $\langle b|u\rangle$ and covariance $\langle Qu|u\rangle$. Hence $\langle u|X(t)\rangle = W(\Theta(t))$ and consequently its a variance gamma process.

Let Y be any HVG-process with parameters (a, b, Q). Then Remark 2.2 yields that X and Y have the same distribution and hence they have the same moments. Since X and Y have the same characteristic function they have the same characteristics. $\square$

APPENDIX A.

A.1. **Properties of Hilbert space valued Lévy processes.**

Lemma A.1. *Let Y be an H-valued process. Let (b_1, Q_1, ν_1) and (b_2, Q_2, ν_2) both be characteristics of Y. Then $b_1 = b_2$, $Q_1 = Q_2$ and $\nu_1 = \nu_2$. In other words, the process Y has exactly one characteristic.*

Proof. Define

$$\varphi_k : H \to \mathbb{C}, u \mapsto i\langle b_k | u \rangle - \frac{1}{2}\langle Q_k u | u \rangle + \int_H (e^{i\langle u | x \rangle} - 1 - i\langle u | x 1_{\{|x| \leq 1\}}\rangle)\nu(dx)$$

for $k \in \{1, 2\}$. [14, Theorem 4.27] yields

$$\exp(\varphi_1(u)) = \mathrm{E}(e^{i\langle u | Y(1) \rangle}) = \exp(\varphi_2(u))$$

for any $u \in H$. In particular, we have $\varphi_1 = \varphi_2$.

Let $u \in H$ and define $\kappa : \mathbb{R} \to H, t \mapsto tu$ and $\psi := \varphi_1 \circ \kappa = \varphi_2 \circ \kappa$. Then ψ is the Lévy exponent of the $\mathbb{R}$-valued Lévy process $\langle Y | u \rangle$. [7, Lemma II.2.44] yields $\langle b_1 | u \rangle = \langle b_2 | u \rangle$, $\langle Q_1 u | u \rangle = \langle Q_2 u | u \rangle$ and $\nu_1^{\langle u | \cdot \rangle} = \nu_2^{\langle u | \cdot \rangle}$. Since this is true for any $u \in H$ we have $b_1 = b_2$. Q_1, Q_2 are positive matrices and hence

$$\langle Q_1 u | v \rangle = \frac{1}{2}\left(\langle Q_1 u | u \rangle + \langle Q_1 v | v \rangle\right) = \langle Q_2 u | v \rangle$$

for any $u, v \in H$. Hence $Q_1 = Q_2$. [8, page 38] yields that the Borel-σ-algebra on H coincides with the cylindrical σ-algebra, i.e. the σ-algebra generated by the continuous linear functionals. Since $\nu_1^{\langle u | \cdot \rangle} = \nu_2^{\langle u | \cdot \rangle}$ for any $u \in H$ they coincide on the cylindrical σ-algebra. Thus $\nu_1 = \nu_2$. $\square$

Definition A.2. A function $g : H \to \mathbb{R}_+$ is called *submultiplicative* if $g(x + y) \leq ag(x)g(y)$ for some constant $a > 0$, cf. [17, Definition 25.2]. A function $f : H \to \mathbb{R}_+$ is called *subadditive* if $f(x + y) \leq f(x) + f(y)$.

Lemma A.3. *Let $g : H \to \mathbb{R}_+$ be a submultiplicative function which is bounded on a neighbourhood of zero. Then there are $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $g(x) \leq c_1 \exp(c_2 |x|)$ for all $x \in H$.*

Proof. This proof is along the lines of [17, Lemma 25.5]. W.l.o.g. assume that $a \geq 1$. Let $y \in H$ and $n \in \mathbb{N}$ then applying submultiplicativity n times we get

$$g(yn) \leq a^{n-1}g(y)^n.$$

If $g(0) = 0$, then $g(y) \leq ag(y)g(0) = 0$ for any $y \in U$ and the claim follows. W.l.o.g. we may assume that $g(0) \neq 0$. Let $\epsilon > 0$ such that g is bounded by $c_1 \geq 1$ on the set $\{x \in H : |x| < \epsilon\}$. Let $y \in H$ and $n \in \mathbb{N}$ such that $|y|/n \leq \epsilon \leq |y|/(n-1)$. Then

$$\begin{aligned}
g(y) &\leq a^{n-1}(g(y/n))^n \\
&\leq (ac_1)^{|y|/\epsilon}c_1 \\
&= c_1 \exp(|y| \log(ac_1)/\epsilon).
\end{aligned}$$

The Lemma follows. □

Remark A.4. Let α, β be finite Borel measures on H. Recall that the convolution $\alpha * \beta$ of the measures α, β is a finite Borel measure on H which is defined by

$$(\alpha * \beta)(B) := \int_H \beta(B - x)\alpha(dx) \quad B \in \mathcal{B}(H)$$

for any $B \in \mathcal{B}(H)$. Moreover, α^{*0} is definedy to be the dirac-measure in 0 and $\alpha^{*n+1} := \alpha * (\alpha^{*n})$ for all $n \in \mathbb{N}$. The total mass of the measure α^{*n} is given by

$$\alpha^{*n}(H) = (\alpha(H))^n.$$

Let $t \geq 0$. Then $\gamma_n := \sum_{k=0}^n \frac{(t\alpha)^{*n}}{n!}$ converges w.r.t. the total variation norm on the space of signed measures of finite total variation to a measure which we denote by $\exp(t\alpha)$. The formula above yields $(\exp(t\alpha))(H) = \exp(t\alpha(H))$ and hence *the convolution semigroup generated by* α which is given by

$$\mu_t := \exp(-t\alpha(H)) \exp(t\alpha) \quad t \geq 0$$

is a probability measure such that $\mu_t * \mu_s = \mu_{t+s}$ for any $s, t \geq 0$.

Lemma A.5. *Let ν_1 be a finite Borel measure on H and define*

$$\mu_t := \exp(-t\nu_1(H)) \exp(t\nu_1)$$

for any $t \in \mathbb{R}_+$. Let g be submultiplicative with constant a. Then

$$\exp(-t\nu_1(H)) \int_H g(x)\nu_1(dx) \leq \int_H g(x)\mu_t(dx) \leq \exp\left(ta \int_H g(x)\nu_1(dx) - t\nu(H)\right)$$

for any $t > 0$. In particular, $\int_H g(x)\nu_1(dx) < \infty$ if and only if $\int_H g(X)\mu_t(dx) < \infty$ for some (and hence all) $t > 0$.

Proof. The first inequality is trivial. Let $t > 0$. Then

$$\int_H g(x)t^n\nu_1^n(dx) \leq a^{n-1}\left(t \int_H g(x)\nu_1(dx)\right)^n \leq \left(ta \int_H g(x)\nu_1(dx)\right)^n.$$

Thus

$$\int_H g(x)\mu_t(dx) \leq \exp\left(ta \int_H g(x)\nu_1(dx) - t\nu_1(H)\right).$$

□

Lemma A.6. *Assume that L has bounded jumps. Let g be submultiplicative and bounded on a neighbourhood of zero. Then*

$$Eg(L(t)) < \infty$$

for any $t \geq 0$.

Proof. Lemma A.3 yields that there are $c_1 > 0$, $c_2 \in \mathbb{R}$ such that $g(x) \leq c_1 \exp(c_2|x|)$. [14, Theorem 4.4] yields

$$Eg(L(t)) \leq c_1 E(e^{c_2|L(t)|}) < \infty.$$

□

The next Proposition does not assume any local boundedness as in [17, Theorem 25.3]. However, it already follows from the Proof of [17, Theorem 25.3] that the boundedness is not needed for the next Proposition in the finite dimensional case.

Proposition A.7. *Let $t > 0$ and g be submultiplicative and measurable and assume that $Eg(L(t)) < \infty$. Then $\int_{\{|x|>1\}} g(x)\nu(dx) < \infty$.*

Proof. This proof is along the lines of the proof of [17, Theorem 25.3].

[14, Theorem 4.23] yields that $L = L_1 + L_2$ where L_1 and L_2 are independent Lévy processes, L_1 has jumps bounded by 1 and L_2 is a compound Poisson process with Lévy measure $\nu_2(B) := \nu(B \cap \{|x| > 1\})$. Let μ_1 be the distribution of $L_1(t)$ and μ_2 be the distribution of $L_2(t)$. Since L_2 is a compound Poisson process we have $\mu_2 = \exp(-t\nu_2(H))\exp(t\nu_2)$. Moreover, we have

$$\int_H \int_H g(x+y)\mu_2(dy)\mu_1(dx) = Eg(L(t)) < \infty.$$

Thus there is $x \in H$ such that

$$\int_H g(x+y)\mu_2(dy) < \infty.$$

Hence Lemma A.5 yields

$$\int_{\{|x|>1\}} g(y)\nu(dy) = \int_H g(y)\nu_2(dy) \le ag(-x)\int_H g(x+y)\nu_2(dy) < \infty.$$

$\square$

Now we generalise [17, Theorem 25.3] to Hilbert space valued Lévy processes.

Theorem A.8. *Let g be submultiplicative, bounded and measurable on a neighbourhood of zero. Then $\int_{\{|x|>1\}} g(x)\nu(dx) < \infty$ if and only if $\mathrm{E}g(L(t)) < \infty$ for some (and hence all) $t > 0$.*

Proof. Proposition A.7 yields the only if part.

[14, Theorem 4.23] yields that $L = L_1 + L_2$ where L_1 and L_2 are independent Lévy processes, L_1 has jumps bounded by 1 and L_2 is a compound Poisson process with Lévy measure $\nu_2(B) := \nu(B \cap \{|x| > 1\})$. Moreover, $\mathrm{E}(g(L(t))) \le \mathrm{E}g(L_1(t))\mathrm{E}g(L_2(t))$ where the first factor is finite by Lemma A.6. Lemma A.5 yields that $\mathrm{E}g(L_2(t)) < \infty$ because

$$\int_H g(x)\nu_2(dx) = \int_{\{|x|>1\}} g(x)\nu(dx) < \infty.$$

Thus $\mathrm{E}g(L(t)) < \infty$. $\square$

Definition A.9. The *growth function* of the process L is the function

$$f : \mathbb{R}_+^d \to \mathbb{R}_+ \cup \{\infty\}, t \mapsto \mathrm{E}|L(t)|.$$

Remark A.10. Let f be the growth function of L.

If L is integrable, then

- f is continuous,
- $f(\theta_1 + \theta_2) \le f(\theta_1) + f(\theta_2)$ for any $\theta_1, \theta_2 \in \mathbb{R}_+^d$,

- $f(\theta_1) \leq f(\theta_1 + \theta_2)$ for any $\theta_1, \theta_2 \in \mathbb{R}_+^d$ and
- $f(\theta) < \infty$ for any $\theta \in \mathbb{R}_+$.

If f_j is the growth function of L_j, then we have $f(\theta) \leq \sum_{j=1}^d f_j(\theta_j) \leq \sqrt{d} f(\theta)$ for any $\theta \in \mathbb{R}_+^d$.

Lemma A.11. *Let f be the growth function of L and let $f(\Theta(1))$ be integrable. Then X is integrable.*

Proof. We have

$$\begin{aligned}
\mathrm{E}|X(1)| &= \mathrm{E}\left(\mathrm{E}\left(|L(\Theta(1))|\big|\Theta\right)\right) \\
&= \mathrm{E}(f(\Theta(1))) \\
&< \infty.
\end{aligned}$$

$\square$

Proposition A.12. *Let $f : H \to \mathbb{R}_+$ be bounded in a neighbourhood of zero and subadditive, i.e. $f(x + y) \leq f(x) + f(y)$. Then $f(L(t))$ is integrable for some (and hence all) $t > 0$ if and only if*

$$\int_{\{|x|>1\}} f(x)\nu(dx) < \infty.$$

In particular, $\mathrm{E}|L(1)| < \infty$ if and only if

$$\int_{\{|x|>1\}} |x|\nu(dx) < \infty.$$

Proof. Define $g(x) := 2 \vee f(x)$ for any $x \in H$. Let $x, y \in H$ such that $f(x) \leq f(y)$. Then

$$\begin{aligned}
g(x + y) &\leq g(x) + g(y) \\
&\leq 2g(y) \\
&\leq g(x)g(y).
\end{aligned}$$

Thus g is submultiplicative and bounded in a neighbourhood of zero. Theorem A.8 yields $\mathrm{E}g(L(1)) < \infty$ if and only if $\int_{\{|x|>1\}} g(x)\nu(dx) < \infty$. The claim follows. $\square$

Proposition A.13. *Let Y be an H-valued stochastic process with independent increments such that*

- *$\langle u|Y \rangle$ is a Lévy process for every $u \in H$.*

Then Y is a Lévy process in law.

Remark A.14. The authors do not know if property (1) in the assumption above is obsolete.

Proof. Define $f : \mathbb{R}_+ \times H \to \mathbb{C}, (t, u) \mapsto \mathrm{E}\exp(i\langle u|Y(t)\rangle)$. Then $f(t, \cdot)$ is the characteristic function of $Y(t)$ for any $t \geq 0$. Moreover, $f(t, u) = \exp(t\rho(u))$ for some function $\psi : H \to \mathbb{C}$ because $\langle u|Y(t)\rangle$ is a Lévy process and $\rho(u)$ is its Lévy exponent evaluated at 1. Since $f(t, \cdot)$ is continuous for any $t > 0$ we conclude that ρ is continuous. Thus f is contiuous. Consequently, Y is stochastically continuous. Let $t, h \in \mathbb{R}_+$. Then

$$\mathrm{E}\exp(i\langle u|Y(t + h) - Y(h)\rangle) = \mathrm{E}\exp(i(\langle u|Y\rangle(t + h) - \langle u|Y\rangle(h))) = f(t, u)$$

for any $t \in \mathbb{R}_+$, $u \in H$ because $\langle u|Y \rangle$ is a Lévy process. Thus Y has stationary incre-ments. $\qquad \square$

A.2. **Estimates.**

Lemma A.15. *Let $N_1, \ldots, N_d$ be compound Poisson processes on $H_1, \ldots, H_d$ and $g : H \to \mathbb{R}_+$ be measurable and subadditive with $g(0) = 0$. Let $N := (N_1, \ldots, N_d)$ be a Lévy process. Then*

$$Eg(N(\theta)) \leq |\theta| \sqrt{d} \int_H g(x) \mu(dx)$$

for any $\theta \in \mathbb{R}_+^d$ where μ is the Lévy measure (or jump intensity measure) of N.

Proof. Let $j \in \{1, \ldots, d\}$, $\theta \in \mathbb{R}_+^d$ and define $g_j(x) := g(\eta_j(x))$ for any $x \in H_j$. [14, Definition 4.14] states that $P^{N_j(\theta_j)} = e^{-\lambda_j \theta_j} \exp(\theta \mu_j)$ where $\lambda_j := \mu_j(H_j)$ and μ_j is the Lévy measure of N_j. Moreover,

$$
\begin{aligned}
\int_{H_j} g_j(x)(\mu_j^{*k})(dx) &\leq \int_{H_j} \cdots \int_{H_j} (g_j(x_1) + \cdots + g_j(x_k)) \mu_j(dx_1) \ldots \mu_j(dx_k) \\
&= \int_{H_j} g_j(x) \mu_j(dx) k \lambda_j^{k-1}.
\end{aligned}
$$

for any $k \in \mathbb{N}$. Thus

$$\int_{H_j} g_j(x)(\exp(\theta_j \mu_j))(dx) \leq \theta_j \int_H g_j(x) \mu_j(dx) e^{\lambda_j \theta_j}$$

and hence

$$Eg_j(N_j(\theta_j)) = \int_{H_j} g_j(x) P^{L_j(\theta_j)} \leq \theta_j \int_{H_j} g_j(x) \mu_j(dx).$$

Since $\int_{H_j} g_j(x) \mu_j(dx) \leq \int_H g(x) \mu(dx)$ the assertet inequality follows. $\qquad \square$

Remark A.16. The inequality in the Lemma above is sharp. Indeed, if $N_1 = \cdots = N_d$ are the same Poisson process with intensity 1, $g : \mathbb{R}^d \to \mathbb{R}, x \mapsto \sum_{j=1}^d |x_j|$ and $\theta = (1, \ldots, 1)$, then

$$\mathrm{E}g(N(\theta)) = d = \sqrt{d}|\theta| \int_H g(x)\mu(dx).$$

Lemma A.17. *Let $M_1, \ldots, M_d$ be a mean zero and square integrable Lévy processes on $H_1, \ldots, H_d$ respectively such that $M := (M_1, \ldots, M_d)$ is a Lévy process. Let $\alpha \in (0, 2]$. Then there is a constant $C > 0$ such that*

$$\mathrm{E}(|M(\theta)|^\alpha) \leq |\theta|^{\alpha/2} C$$

for any $\theta \in \mathbb{R}_+^d$. Moreover, the constant C can be chosen as

$$\left(\mathrm{Tr}(\Gamma) + \int |x|^2 \mu_j(dx) \right)^{\alpha/2}$$

where (β, Γ, μ) is the characteristics of M.

Proof. The case $\alpha = 2$ follows from the Pythagorean theorem. Indeed,

$$
\begin{aligned}
\mathrm{E}(|M(\theta)|^2) &= \sum_{j=1}^{d} \mathrm{E}(|M_j(\theta_j)|^2) \\
&= \sum_{j=1}^{d} \theta_j \mathrm{E}(|M_j(1)|^2) \\
&\leq |\theta| \mathrm{E}(|M(1)|^2) \\
&= |\theta| C
\end{aligned}
$$

for any $\theta \in \mathbb{R}_+^d$ where $C := \mathrm{E}(|M(1)|^2)$.

The other cases follow from the case $\alpha = 2$ by Jensens inqueality. Indeed, we have

$$
\begin{aligned}
\mathrm{E}(|M(\theta)|^\alpha) &= (\mathrm{E}(|M(\theta)|^2))^{\alpha/2} \\
&\leq (|\theta| C)^{\alpha/2} \\
&= |\theta|^{\alpha/2} C^{\alpha/2}.
\end{aligned}
$$

for any $\alpha \in (0, 2]$ and any $\theta \in \mathbb{R}_+^d$. $\qquad\square$

Lemma A.18. *Let L be integrable. Then there are constants $C_1, C_2 \in \mathbb{R}_+$ such that*

$$
\mathrm{E}\left(|L(\theta)|\right) \leq |\theta| C_1 + |\theta|^{1/2} C_2
$$

for any $\theta \in \mathbb{R}_+^d$.

Proof. Define $g := |\cdot|$. Then g is subadditive. [14, Theorem 4.23] implies that $L(\theta) = a\theta + M(\theta) + N(\theta)$ for some $a \in H$ a mean zero and square integrable Lévy process M and a compound Poisson process N where the Lévy measure of N is given by $\mu(A) = \nu(A \cap \{x \in H : |x| > 1\})$ for any $A \in \mathcal{B}(H)$. The two previous Lemmas yield

$$
\begin{aligned}
\mathrm{E}g(L(\theta)) &\leq g(a\theta) + \mathrm{E}g(N(\theta)) + \mathrm{E}g(M(\theta)) \\
&\leq |\theta||a| + |\theta| \int_{|x|>1} g(x)\mu(dx) + |\theta|^{1/2} C_2
\end{aligned}
$$

for some constant $C_2 > 0$ and any $\theta \in \mathbb{R}_+^d$. [14, Proposition 4.18] yields that $C_3 := \int_{|x|>1} g(x)\mu(dx) < \infty$. Define $C_1 := |a| + C_3$. Then

$$
\mathrm{E}(|L(\theta)|) \leq |\theta| C_1 + |\theta|^{1/2} C_2
$$

as claimed. $\qquad\square$

We now state some technical Lemmas which are needed for the proof of Theorem 2.4. The first one essentially states that $\mathbb{R}_+^d \to \mathbb{R}, \theta \mapsto \mathrm{E}f(L(\theta))$ growth at most linearly for smooth functions f. The second one states that this is also true for $\theta \mapsto \mathrm{E}\chi(L(\theta))$.

Lemma A.19. *Let $f : H \to \mathbb{R}$ be bounded and uniformly continuous such that its derivatives up to order two are also bounded and uniformly continous. Then there is a constant $C > 0$ such that*

$$
|\mathrm{E}(f(L(\theta))) - f(0)| \leq \sum_{j=1}^{d} \theta_j C
$$

for any $\theta \in \mathbb{R}_+^d$. Moreover, the constant C can be chosen as

$$\sup_{x \in H} 2|f(x)|\nu(|x| > 1) + |b|\|Df(x)\|_{H'} + \frac{1}{2}\left(\mathrm{Tr}(Q) + \int_{\{|x| \leq 1\}} |x|^2 \nu(dx)\right)\|D^2 f(x)\|_{\mathrm{op}}$$

where $\| \cdot \|_{H'}$ denotes the operator norm on the space of linear functionals from H to $\mathbb{R}$ and $\| \cdot \|_{\mathrm{op}}$ denotes the operator norm on the space of linear functions on H.

Proof. We first show the inequality for $d = 1$. Let UC_b^2 be the set of functions which are bounded and uniformly continuous and whose derivatives up to order two are also bounded and uniformly continous. [14, Theorem 5.4] yields that L is a Markov process and the domain of its generator $\mathcal{A}$ contains UC_b^2 and

$$\mathcal{A}f(x) = \langle b, Df(x)\rangle + \frac{1}{2}\mathrm{Tr}(Q)D^2 f(x) + \int_H (f(x+y) - f(x) - 1_{\{|y| \leq 1\}}\langle y, Df(x)\rangle)\nu(dy)$$

for any $x \in H$. In particular,

$$M(t) := f(L(t)) - \int_0^t \mathcal{A}f(L(s))ds$$

is a martingale w.r.t. the filtration generated by L. Define $C := \sup_{\{x \in H\}} \mathcal{A}f(x)$. The inequality follows.

For arbitrary d the inequality follows from a simple induction. $\square$

Lemma A.20. *There is a constant $C > 0$ such that*

$$|\mathrm{E}\chi(L(\theta))| \leq |\theta|C$$

for any $\theta \in \mathbb{R}_+^d$.

Proof. Let $\chi_1 : H \to H$ such that χ_1 is twice continuously differentiable, its support is contained in the centered ball of radius 2, $\chi_1(x) = x$ for $x \in H$ with $|x| \leq 1$ and its first two derivatives vanish on its zeros except for the zero in 0. Then the restriction f of $|\cdot| \circ \chi_1$ to the set $\{x \in H : |x| \geq 1\}$ has a twice continuously differentiable continuation $\chi_2 : H \to \mathbb{R}_+$ such that χ_2 vanishes on the centered ball of radius 0.5. By Lemma A.19 for each $\alpha \in H'$ there is a constant C_α such that

$$|\mathrm{E}(\alpha \circ \chi_1)(\theta)| \leq C_\alpha \sum_{j=1}^d \theta_j \leq C_1 \sqrt{d}|\theta|$$

for any $\theta \in \mathbb{R}_+^d$ where C_1 is defined by

$$\sup_{x \in H}\left(2|\chi_1(x)|\nu(|x| > 1) + |b|\|D\chi_1(x)\|_{op}\right.$$

$$\left. + \frac{1}{2}\left(\mathrm{Tr}(Q) + \int_{\{|x| \leq 1\}} |x|^2 \nu(dx)\right)\|D^2\chi_1(x)(y, z)\|_{B^2}\right)$$

and where $\| \cdot \|_{\mathrm{op}}$ denotes the operator norm on the space of linear functions on H and $\| \cdot \|_{B^2}$ denotes the operator norm of bilinear functions from $H \times H$ to H. For $\theta \in$

$\mathbb{R}_+^d$ the Hahn-Banach theorem [15, Theorem 3.2] yields that there is $\alpha_\theta \in H'$ such that $\|\mathrm{E}\chi_1(L(\theta))\| = \alpha_\theta(E\chi_1(L(\theta)))$ and $\|\alpha_\theta\|_{op} = 1$. Hence

$$\|\mathrm{E}\chi_1(L(\theta))\| \le C_1\sqrt{d}|\theta|.$$

Lemma A.19 yields that there is a constant C_2 such that $\mathrm{E}\chi_2(L(\theta)) \le |\theta|C_2$. Then

$$
\begin{aligned}
|\mathrm{E}\chi(L(\theta))| &\le |\mathrm{E}\chi_1(L(\theta))| + |\mathrm{E}(\chi_1 - \chi)(L(\theta))| \\
&\le C_1\sqrt{d}|\theta| + \mathrm{E}\chi_2(L(\theta)) \\
&\le C|\theta|
\end{aligned}
$$

where $C := \sqrt{d}C_1 + C_2$. $\qquad\square$

Lemma A.21. *Let $u \in H$ and*

$$g : \mathbb{R}_+^d \to \mathbb{C}, u \mapsto \mathrm{E}\left|e^{i\langle u|L(\theta)\rangle} - 1 - i\langle u|\chi(L(\theta))\rangle\right|.$$

Then there is a constant $C > 0$ such that $|g(\theta)| \le |\theta|C$ for any $\theta \in \mathbb{R}_+^d$.

Proof. Let $\chi_1 : H \to H$ such that χ_1 is twice continuously differentiable, its support is contained in the centered ball of radius 2, $\chi_1(x) = x$ for $x \in H$ with $|x| \le 1$ and its first two derivatives vanish on its zeros except for the zero in 0. In the proof of Lemma A.20 we have shown that there is a constant C_2 such that

$$\mathrm{E}|\chi_1(L(\theta)) - \chi(L(\theta))| \le |\theta|C_2.$$

Let

$$f : H \to \mathbb{C}, x \mapsto |e^{i\langle u|x\rangle} - 1 - i\langle u|\chi_1(x)\rangle|.$$

Lemma A.19 yields $\mathrm{E}f(L(\theta)) \le |\theta|C_1$ for some constant $C_1 > 0$. Thus

$$g(\theta) \le \mathrm{E}f(L(\theta)) + |u|\mathrm{E}|\chi_1(L(\theta)) - \chi(L(\theta))| \le (C_1 + |u|C_2)|\theta|$$

for any $\theta \in \mathbb{R}_+^d$. $\qquad\square$

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(Fred Espen Benth), CENTRE OF MATHEMATICS FOR APPLICATIONS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, N–0316 OSLO, NORWAY

E-mail address: fredb@math.uio.no
URL: http://folk.uio.no/fredb/

(Paul Krühner), DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, N–0316 OSLO, NORWAY

E-mail address: paulkru@math.uio.no